Cantor set and measure through matrix eigenvector v.1

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We have defined the Cantor set through an iterated function system consisting of two transformations that are selected with equal probabilities. These transformations, let us call them here T_1, T_2 :

$$T_1(x) = \frac{1}{3}x, \tag{1}$$

$$T_2(x) = \frac{1}{3}x + \frac{2}{3}. \tag{2}$$

act on points of the interval $[0, 1]$. Whenever we have a transformation acting on points, it induces transformation of functions, we denote it by T^* :

$$(T^*f)(x) = f(T(x)). \tag{3}$$

When we have two transformations, T_1 and T_2, that are selected with probabilities p_1 and p_2, the resulting function is also weighted with probabilities. This way we arrive at what is called the Koopman operator T associated with the IFS:

$$(T^*f)(x) = p_1 f(T_1(x)) + p_2 f(T_2(x)). \tag{4}$$

Dual to the space of functions is the space of bounded measures. Given a measure μ, we can associate with each function (that is 'measurable' etc.) the number $(\mu, f) = \int f\, d\mu$. We define the dual operator T_* on measures by the formula:

$$(T_*\mu, f) = (\mu, T^*f). \tag{5}$$

Then T_* is called the Frobenius-Perron operator associated with the IFS. We are looking for a probabilistic measure that is invariant with respect to T_*. In most cases studied in the literature one can prove that such a measure exists and is unique. For the Cantor system we can calculate and graphically represent an approximation for this invariant measure. To this end we discretize the interval $[0, 1]$ into, say, $N = 1000$ small intervals $\Delta_i = [(i-1) * \Delta, i * \Delta]$, $i = 1, ..., N$, $\Delta = 1.0/N$. In the space of function we chose an orthogonal basis $e_i(x) = \chi_{\Delta_i}(x)$, where χ_Δ denotes the characteristic function of the set $\Delta \subset [0, 1]$: $\chi_\Delta(x) = 1$ for $x \in \Delta$, otherwise $\chi_\Delta(x) = 0$.. Then we approximate the operator T^* by a finite dimensional. Namely, we want to decompose $T^* e_i$, projected onto the subspace generated by e_i,into e_j :

$$T^* e_i = \sum_j T^*_{ji} e_j. \tag{6}$$

In order to calculate the matrix coefficients T^*_{ji}, we take scalar products (in L^2) of the above formula with e_k.

$$(e_k, T^* e_i) = \sum_j T^*_{ji}(e_k, e_j). \tag{7}$$

The functions e_i are orthogonal and $(e_k, e_j) = \Delta \delta_{k,j}$, where δ_{kj} is the Kronecker *delta*. This way we get the formula:

$$T^*_{ki} = \frac{1}{2\Delta}(e_k, T^* e_i) = \frac{1}{2\Delta} \int_{\Delta_k} \left(\chi_{\Delta_i}(\frac{x}{3}) + \chi_{\Delta_i}(\frac{x+2}{3}) \right) dx, \tag{8}$$

where the factor 2 in the denominator comes from the fact that each of the two transformations is selected with the probability 1/2.

Now, $\chi_\Delta(\frac{x}{3}) = \chi_{3\Delta}(x)$, and $\chi_\Delta(\frac{x+2}{3}) = \chi_{3\Delta-2}(x)$. Therefore we obtain the following formula:

$$T^*_{ki} = \frac{1}{2\Delta}(|3\Delta_i \cap \Delta_k| + |(3\Delta_i - 2) \cap \Delta_k|, \tag{9}$$

where we denoted by $|.|$ the length of the corresponding interval.

Notice that owing to the fact that $\sum_i \Delta_i$ is the whole interval $[0, 1]$, we get $\sum_i T^*_{ki} = 1$. Thus the sum of elements in every row of the matrix T^*_{ki} is one, therefore it has an eigenvector belonging to the eigenvalue 1 (the vector with all its components equal 1.). But a matrix and a transposed matrix have the same eigenvalues. The Frobenius Perron operator is dual to T^*, therefore it is represented by the transposed matrix, let us call it $T : T_{ik} = T_{ki}$. It's eigenvector to the eigenvalue one is exactly the eigenvector we are looking for - our approximation to the invariant measure.

Given two intervals (a_1, b_1) and (c_1, d_1) we have the following formula for the length of their intersection:

$$|(a_1, b_1) \cap (a_2, b_2)| = max\,(0, min(b_1, b_2) - max(a_1, a_2))\,. \qquad (10)$$

Applying to our case we obtain:

$$\begin{aligned} T_{ik} = \frac{1}{2}\,&(\max(0,\, \min(3i, k) - \max(3i-3, k-1))) \\ &+ \frac{1}{2}\max\,(0,\, \min(3i-2N, k) - \max(3i-3-2N, k-1)))\,. \end{aligned} \qquad (11)$$

The matrix T has a simple band structure, especially regular when $N = 3k$, where k is an integer. Fig. 1 is a graphical representation of this structure for $k = 18$. We have there either 0 or $\frac{1}{2}$. The problem of finding the invariant eigenvector of such a matrix can be solved exactly. For $N = 1000$ the solution can be found numerically - it is represented on Fig. 2.

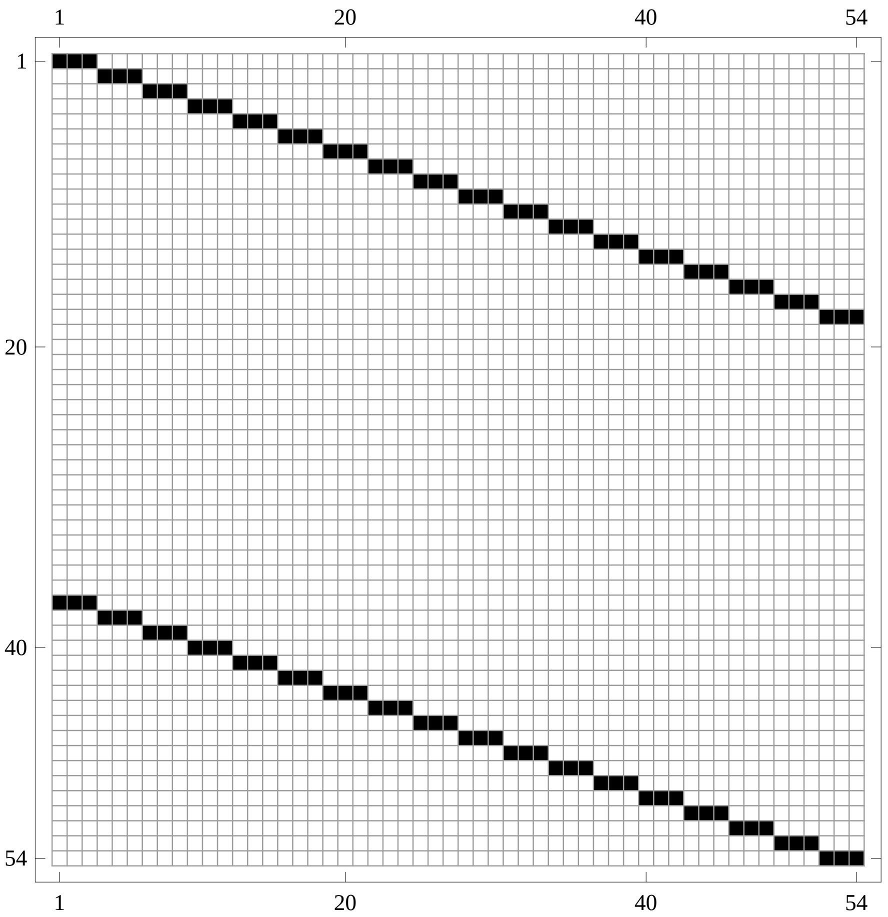

Figure 1: Band structure of the Frobenius-Perron matrix for the Cantor set for $N = 54$.

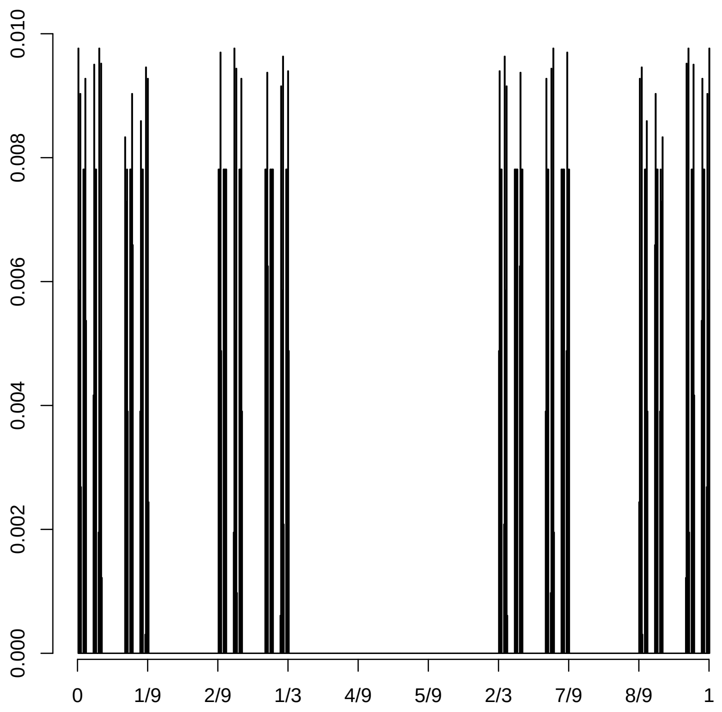

Figure 2: Frobenius-Perron matrix for the Cantor set for $N = 1000$.